

LATTICE COMPLEMENTS AND THE SUBADDITIVITY OF SYZYGIES OF SIMPLICIAL FORESTS

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Abstract

We prove the subadditivity property for the maximal degrees of the syzygies of facet ideals simplicial forests. For such an ideal I , if the i -th Betti number is nonzero and $i = a + b$, we show that there are monomials in the lcm lattice of I that are complements in part of the lattice, each supporting a nonvanishing a -th and b -th Betti numbers. The subadditivity formula follows from this observation.

1 Introduction

Let $= k[x_1, \dots, x_n]$, where k is a field, and let I be a graded ideal of S , and suppose S/I has minimal graded free resolution

$$0 \rightarrow \oplus_{j \in \mathbb{N}} S(-j)^{\beta_{p,j}} \rightarrow \oplus_{j \in \mathbb{N}} S(-j)^{\beta_{p-1,j}} \rightarrow \dots \rightarrow \oplus_{j \in \mathbb{N}} S(-j)^{\beta_{1,j}} \rightarrow S$$

where the graded Betti numbers $\beta_{i,j}(S/I)$ denote the rank of the degree j component $S(-j)$ appearing in the i th homological degree in this sequence.

For an integer i , define

$$t_a(I) = \max\{j \mid \beta_{a,j}(S/I) \neq 0\}.$$

We say that the degrees of the Betti numbers of I satisfy the *subadditivity property* if

$$t_{a+b}(I) \leq t_a(I) + t_b(I)$$

for all $a, b > 0$ with $a + b \leq p$, where p is the projective dimension of S/I .

It is known that in general the subadditivity property does not hold (Avramov, Conca, Iyengar [ACI]), but under restrictive conditions, many cases have been known to hold. These include some algebras of krull dimension at most 1 (Eisenbud, Huneke and Ulrich [EHU]), when $a = 1$ (Fernández-Ramos and Gimenez [FG] if I is generated by degree 2 monomials, Herzog and Srinivasan [HS] when b is the projective dimension of S/I or when I is any monomial ideal), in certain homological degrees in the case of Gorenstein algebras (El Khoury and Srinivasan [ES]) and the

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case $a = 1, 2, 3$ for a monomial ideal generated in degree 2 (Abdelfatah and Nevo [AN]). Otherwise, the question is wide open for the class of monomial ideals.

We approach this problem using “lattice complements” which appear in the topology of lattices. Two elements of a lattice are complements if their join and their meet are $\hat{1}$ and $\hat{0}$, respectively. In the case of two monomials in the lcm lattice of a monomial ideal, they are complements if their gcd is not in the ideal and their lcm is the lcm of all the generators.

Our motivation for using complements is the fact that if the top degree Betti number of a monomial ideal I is nonzero, then every monomial in the lcm lattice of I has a complement. This follows from the interpretation of Betti numbers of monomial ideals in terms of homology of open intervals in lattices by Gasharov, Peeva and Welker ([GPW, P]) and Baclawski’s ([B]) work that relates the homology of lattices to the existence of complements.

By polarization, to find Betti numbers of monomial ideals it is enough to consider Betti numbers of square-free monomial ideals. Moreover, in this case inquiries about a specific graded Betti number reduces to that of “top degree” Betti numbers – see below for more on this.

So we ask the following question.

Question 1.1. If I is a square-free monomial involving n variables and $\beta_{i,n}(S/I) \neq 0$, $a, b > 0$ and $i = a + b$, are there complements \mathbf{m} and \mathbf{m}' in the lcm lattice of I with nonzero multigraded Betti numbers $\beta_{a,\mathbf{m}}(S/I)$ and $\beta_{b,\mathbf{m}'}(S/I)$?

If the answer is positive, then the subadditivity conjecture is true for all monomial ideals, since $\deg(\mathbf{m}) + \deg(\mathbf{m}') \geq n$.

In this paper we give a positive answer to this question in the case where I is the facet ideal of a simplicial forest Δ . In this case, we take advantage of the a recursive formula relating Betti numbers in each homological degree to lower ones [HV, F2]. What we really use is the inductive existence of a facet with a free vertex, which acts as a splitting facet (in the Eliahou-Kervaire sense). We hope, however, that some of these methods can be used to study the general version of the subadditivity property for monomial ideals. A further study to see if Question 1.1 holds in general would be extremely useful for this purpose.

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2 Setup

A **simplicial complex** Δ is a set of subsets of a set A , such that if $F \in \Delta$ then all subsets of F are also in Δ . Every element of Δ is called a **face** of Δ , the maximal elements under inclusion are called **facets** and the **dimension of a face** F of Δ is defined as $|F| - 1$. The faces of dimensions 0 and 1 are called **vertices** and **edges**, respectively, and $\dim \emptyset = -1$. The **dimension of** Δ is the maximal dimension of its facets. We denote the set of vertices of Δ by $V(\Delta)$.

A **subcollection** of Δ is a simplicial complex whose facets are also facets of Δ ; in other words a simplicial complex generated by a subset of the set of facets of Δ . If $u \subseteq V(\Delta)$, then the subcollection $\Delta_{[u]}$ consisting of all facets of Δ contained in u is an **induced subcollection** of Δ .

We denote the set of facets of Δ by $\text{Facets}(\Delta)$. If $\text{Facets}(\Delta) = \{F_1, \dots, F_q\}$, we write $\Delta = \langle F_1, \dots, F_q \rangle$. The simplicial complex obtained by **removing the facet** F_i from Δ is

$$\Delta \setminus \langle F_i \rangle = \langle F_1, \dots, \hat{F}_i, \dots, F_q \rangle.$$

If F is a facet of Δ , then $\overline{F} = V(\Delta) \setminus F$, i.e. all vertices of Δ that are not in F .

A facet F of Δ simplicial complex is called a **leaf** if either F is the only facet of Δ or for some facet $G \in \Delta \setminus \langle F \rangle$ we have $F \cap (\Delta \setminus \langle F \rangle) \subseteq G$. Equivalently, we can say a facet F is a leaf of Δ if $F \cap (\Delta \setminus \langle F \rangle)$ is a face of $\Delta \setminus \langle F \rangle$. It follows immediately from the definition above that a leaf F must contain at least one vertex that belongs to no other facet of Δ but F ; we call such a vertex a **free vertex**.

We call Δ a **simplicial forest** if every nonempty subcollection of Δ has a leaf. A connected simplicial forest is called a **simplicial tree**.

Let $S = k[x_1, \dots, x_n]$ and I a square-free monomial ideal in S . For a subset $u \subset \{x_1, \dots, x_n\}$, we denote by \mathbf{m}_u the **square-free monomial with support** u , that is

$$\mathbf{m}_u = \prod_{x_i \in u} x_i.$$

The **facet complex of** I , denoted is the simplicial complex

$$\mathcal{F}(I) = \langle u \mid \mathbf{m}_u \text{ is a generator of } I \rangle.$$

Conversely, given a simplicial complex Δ on vertices from the set $\{x_1, \dots, x_n\}$, we can define the **facet ideal of** Δ as

$$\mathcal{F}(\Delta) = (\mathbf{m}_u \mid u \text{ is a facet of } \Delta),$$

which is an ideal of S .

One of the properties of simplicial trees that we will be using in this article is the following.

Lemma 2.1 (Localization of a forest is a forest [F1]). *Suppose Δ is a simplicial forest with facet ideal I in the polynomial ring S . Then for any prime ideal p of S , I_p is the facet ideal of a simplicial forest which we denote by Δ_p .*

For a simplicial complex Δ with $I = \mathcal{F}(\Delta) \subseteq S$, by $\beta_{i,j}(\Delta)$ we mean $\beta_{i,j}(S/I)$. The localization property has a substantial effect on the calculation of Betti numbers of forests.

If I is a square free monomial ideal in the polynomial ring $S = k[x_1, \dots, x_n]$ and with facet complex Δ , then every graded Betti number $\beta_{i,j}(S/I)$ is calculated by taking the sum of all *multi-graded* Betti numbers $\beta_{i,\mathbf{m}}(S/I)$ where \mathbf{m} is a square-free monomial in S of degree j (see for example [P]). Such a monomial \mathbf{m} is in fact \mathbf{m}_u for some $u \subseteq \{x_1, \dots, x_n\}$, and using for example the Taylor complex [T, P], one can see that

$$\beta_{i,\mathbf{m}_u}(\Delta) = \beta_{i,j}(\Delta_{[u]}) = \beta_{i,|u|}(\Delta_{[u]})$$

where $\Delta_{[u]}$ is the induced subcollection of Δ on u , and $j = |u|$. The Taylor resolution also shows that if $\beta_{i,\mathbf{m}_u}(\Delta) \neq 0$, then $\Delta_{[u]}$ must have exactly u vertices. These observations reduce the calculation of $\beta_{i,j}(\Delta)$ to the calculation of the “top degree” Betti numbers of certain subcollections (see Remark 2.3 and Lemma 3.1 of [EF1]).

When I is the facet ideal of a simplicial tree Δ , much more about $\beta_{i,j}(\Delta)$ is known, see for example [EF1, EF2]. In particular, a recursive formula for the calculation of the Betti numbers of trees in [F2], which is deduced from a splitting formula due to Hà and Van Tuyl (Theorem 5.5 of [HV]) can be used effectively in this case.

If Δ is a connected simplicial complex with facet ideal I and F is facet of Δ with a free vertex (for example a leaf of a tree), then Hà and Van Tuyl's Theorem 5.5 gives ([F2]), for $i, j \geq 0$

$$\beta_{i,j}(\Delta) = \beta_{i,j}(\Delta \setminus \langle F \rangle) + \beta_{i-1,j-|F|}((\Delta \setminus \langle F \rangle)_{\overline{F}}). \quad (1)$$

In what follows, we will use the localized complex $\Gamma = (\Delta \setminus \langle F \rangle)_{\overline{F}}$ extensively. It is worth observing that if $\Delta = \langle F_1, \dots, F_q \rangle$ then Γ has as facets the minimal elements, under inclusion, of

$$F_1 \setminus F, \dots, F_q \setminus F$$

which correspond to the generators of the ideal $I_{(\overline{F})}$. This leads to the following observation, which was used in the case of trees in [EF2].

Proposition 2.2. *Let Δ be a simplicial complex on n vertices, F a facet of Δ with a free vertex, and suppose $\Gamma = (\Delta \setminus \langle F \rangle)_{\overline{F}}$. Then for every i we have*

$$\beta_{i,n}(\Delta) = \beta_{i-1,n-|F|}(\Gamma).$$

Proof. If Δ is connected, then since F has a free vertex, $\Delta \setminus \langle F \rangle$ has strictly less than n vertices, so in Equation (1) $\beta_{i,n}(\Delta \setminus \langle F \rangle) = 0$, which results in $\beta_{i,n}(\Delta) = \beta_{i-1,n-|F|}(\Gamma)$.

Suppose Δ has connected components $\Delta_1, \dots, \Delta_r$ each with n_1, \dots, n_r vertices, respectively. Assume, without loss of generality, F is a facet of Δ_1 . The connected components of Γ are of the form

$$\Gamma_a = (\Delta_a \setminus \langle F \rangle)_{\overline{F}}$$

where if $a > 1$, one can see immediately that $\Gamma_a = \Delta_a$, as F and Δ_a will have no vertices in common. So we can write (see Lemma 3.2 of [EF1])

$$\begin{aligned} \beta_{i,n}(\Delta) &= \sum_{u_1 + \dots + u_r = i} \beta_{u_1,n_1}(\Delta_1) \dots \beta_{u_r,n_r}(\Delta_r) \\ &= \sum_{u_1 + \dots + u_r = i} \beta_{u_1-1,n_1-|F|}((\Delta_1 \setminus \langle F \rangle)_{\overline{F}}) \beta_{u_2,n_2}(\Delta_2) \dots \beta_{u_r,n_r}(\Delta_r) \\ &= \sum_{u_1 + \dots + u_r = i} \beta_{u_1-1,n_1-|F|}(\Gamma_1) \beta_{u_2,n_2}(\Gamma_2) \dots \beta_{u_r,n_r}(\Gamma_r) \\ &= \beta_{i-1,n-|F|}(\Gamma). \end{aligned}$$

□

3 Betti numbers of complements

Let I be a square-free monomial ideal in $S = k[x_1, \dots, x_n]$. We denote the lcm lattice of I by $\text{LCM}(I)$. The atoms of this lattice are the generators of I and the other members are lcm's of the generators of I ordered by divisibility. The top element $\hat{1}$ is the lcm of all generators of I , and the bottom element $\hat{0}$ is 1. See [P] for more on lcm lattices and their properties.

The following definition is an adaptation of the usual concept of lattice complements to the lcm lattice.

Definition 3.1. Let I be a square-free monomial ideal. Two monomials \mathbf{m} and \mathbf{m}' in $\text{LCM}(I) \setminus \{\hat{0}, \hat{1}\}$ are called **complements** if

- $\text{lcm}(\mathbf{m}, \mathbf{m}') = \hat{1}$ and
- $\text{gcd}(\mathbf{m}, \mathbf{m}') \notin I$.

This definition leads directly to the following statement.

Lemma 3.2. Let Δ be a simplicial complex on n vertices and with facet ideal $I \subseteq S$, and let u and v be two proper subsets of $\{x_1, \dots, x_n\}$, with $\mathbf{m}_u, \mathbf{m}_v \in \text{LCM}(I) \setminus \{\hat{0}, \hat{1}\}$. Then the following are equivalent.

1. \mathbf{m}_u and \mathbf{m}_v are complements in $\text{LCM}(I)$
2. (a) $u \cup v = \{x_1, \dots, x_n\}$, and
(b) the two induced subcollections $\Delta_{[u]}$ and $\Delta_{[v]}$ have no facet in common.

Based on the observation above, we call two proper induced subcollections $\Delta_{[u]}$ and $\Delta_{[v]}$ of Δ **complements in Δ** if the two conditions in Lemma 3.2 (2) hold.

Lemma 3.3. Let Δ be a simplicial complex, F a facet of Δ with a free vertex, $\Gamma = (\Delta \setminus \langle F \rangle)_{\overline{F}}$, and $u \subset \overline{F}$.

1. $\Gamma_{[u]} = (\Delta_{[F \cup u]} \setminus \langle F \rangle)_{\overline{F}}$.
2. If $\Gamma_{[u]}$ has $|u|$ vertices then $\Delta_{[F \cup u]}$ has $|u \cup F| = |u| + |F|$ vertices.

Proof. 1. We show each inclusion.

- (\subseteq) Let $G \setminus F$ be a facet of $\Gamma_{[u]}$, then G is a facet of Δ such that $(G \setminus F) \subseteq u$, which implies that $G \subseteq u \cup F$. Hence G is a facet of $\Delta_{[F \cup u]}$. If H is another facet of $\Delta_{[F \cup u]}$ with $(H \setminus F) \subsetneq (G \setminus F)$, then as $(H \setminus F) \subseteq u$ we will have $(H \setminus F) \in \Gamma_{[u]}$ which contradicts the fact that $G \setminus F$ is a facet of $\Gamma_{[u]}$.
- (\supseteq) Suppose $G \setminus F$ is a facet of $(\Delta_{[F \cup u]} \setminus \langle F \rangle)_{\overline{F}}$. Then $G \subseteq F \cup u$ and hence $(G \setminus F) \subseteq u$. If $(G \setminus F) \notin \Gamma_{[u]}$ then there is another facet H of Δ with $(H \setminus F) \subsetneq (G \setminus F) \subseteq u$. But then $H \setminus F \in (\Delta_{[F \cup u]} \setminus \langle F \rangle)_{\overline{F}}$, which is a contradiction.

2. Suppose $\Gamma_{[u]} = \langle G_1 \setminus F, \dots, G_t \setminus F \rangle$, where by the previous part we can pick G_1, \dots, G_t to be facets of $\Delta_{[F \cup u]}$. Then $|(G_1 \cup \dots \cup G_t) \setminus F| = |u|$ and also $F \in \Delta_{[F \cup u]}$. Since $u \cap F = \emptyset$ we have already accounted for $|F \cup u|$ vertices in $\Delta_{[F \cup u]}$. However this is the maximum number of vertices that $\Delta_{[F \cup u]}$ could have, so our claim is proved. \square

Theorem 3.4. *Let Δ be a simplicial forest with n vertices and more than one facet, and suppose $\beta_{i,n}(\Delta) \neq 0$. Then for every facet G of Δ , $\Delta_{[G]}$ has a complement $\Delta_{[u]}$ in Δ with $\beta_{i-1,|u|}(\Delta_{[u]}) \neq 0$.*

Proof. We use induction on n . The smallest case is $n = 2$ and Δ a complex consisting of two isolated vertices $F = \{x_1\}$ and $G = \{x_2\}$, and we have $\beta_{2,2}(\Delta) \neq 0$ and $\beta_{1,1}(\Delta_{[F]}) = \beta_{1,1}(\Delta_{[G]}) \neq 0$. Since $\Delta_{[F]}$ and $\Delta_{[G]}$ are complements, this settles the statement in the case $n = 2$.

Since every forest has at least two leaves ([F1]), we can choose F to be a leaf of Δ with $F \neq G$. Suppose $\Gamma = (\Delta \setminus \langle F \rangle)_{\overline{F}}$. By Proposition 2.2, $\beta_{i,n}(\Delta) \neq 0$ results in $\beta_{i-1,n-|F|}(\Gamma) \neq 0$, which in particular implies that Γ has $n - |F|$ vertices.

By Lemma 2.1, Γ is a forest, so it satisfies the induction hypothesis.

Now Γ has a facet $H \setminus F$ such that $(H \setminus F) \subseteq (G \setminus F)$ (with $H = G$ possible). By the induction hypothesis, $\Gamma_{[H \setminus F]}$ has a complement $\Gamma_{[v]}$ in Γ such that $\beta_{i-2,|v|}(\Gamma_{[v]}) \neq 0$, which in particular implies that $\Gamma_{[v]}$ has $|v|$ vertices.

By Lemma 3.3, $\Delta_{[F \cup v]}$ has $|F| + |v|$ vertices and

$$\Gamma_{[v]} = (\Delta_{[F \cup v]} \setminus \langle F \rangle)_{\overline{F}}.$$

Proposition 2.2 now implies that $\beta_{i-1,|F \cup v|}(\Delta_{[F \cup v]}) \neq 0$.

We set $u = F \cup v$ and show that $\Delta_{[u]}$ and $\Delta_{[G]}$ are complements in Δ . Since $\Gamma_{[v]}$ and $\Gamma_{[H \setminus F]}$ are complements in Γ ,

$$v \cup (H \setminus F) = V(\Gamma) = (V(\Delta) \setminus F)$$

and add to this the fact that $(H \setminus F) \subseteq (G \setminus F)$ to conclude

$$u \cup G = v \cup F \cup G = V(\Delta).$$

If $\Delta_{[u]}$ and $\Delta_{[G]}$ have a facet in common, then $G \in \Delta_{[u]}$, which implies that $G \subseteq u = v \cup F$. On the other hand

$$(H \setminus F) \subseteq (G \setminus F) \subseteq v \implies (H \setminus F) \in \Gamma_{[v]},$$

which contradicts $\Gamma_{[v]}$ and $\Gamma_{[H \setminus F]}$ being complements in Γ .

So $\Delta_{[u]}$ and $\Delta_{[G]}$ are complements in Δ and we are done. \square

Theorem 3.5. *Let Δ be a simplicial forest with more than one facet and facet ideal $I \subseteq S$. Suppose $\beta_{i,n}(\Delta) \neq 0$, and $i = a + b$ for some positive integers a and b . Then there are complements $\Delta_{[u]}$ and $\Delta_{[w]}$ in Δ with $\beta_{a,|u|}(\Delta_{[u]}) \neq 0$ and $\beta_{b,|w|}(\Delta_{[w]}) \neq 0$.*

Proof. Without loss of generality assume Δ has n vertices.

We prove the statement by induction on n . The base case is $n = 2$, where Δ is two isolated vertices. In this case $i = 2 = 1 + 1$, and the claim follows from Theorem 3.4.

We consider the general case. From Theorem 3.4 the statement is true if $a = 1$ or $b = 1$. So we may assume $a, b > 1$.

Let F be a leaf of Δ and let $\Gamma = (\Delta \setminus \langle F \rangle)_{\overline{F}}$. By Proposition 2.2 $\beta_{i-1, n-|F|}(\Gamma) \neq 0$, and in particular $|V(\Gamma)| = n - |F|$. By the induction hypothesis, since $i - 1 = (a - 1) + b$, there are complements $\Gamma_{[u']}$ and $\Gamma_{[v']}$ in Γ such that

$$\beta_{a-1, |u'|}(\Gamma_{[u']}) \neq 0 \text{ and } \beta_{b, |v'|}(\Gamma_{[v']}) \neq 0.$$

Let $u = u' \cup F$ and $v = v' \cup F$. By Lemma 3.3 we have that

$$\Gamma_{[v']} = (\Delta_{[v]} \setminus \langle F \rangle)_{\overline{F}} \text{ and } \Gamma_{[u']} = (\Delta_{[u]} \setminus \langle F \rangle)_{\overline{F}}$$

and $\Delta_{[u]}$ and $\Delta_{[v]}$ have $|u|$ and $|v|$ vertices, respectively.

Proposition 2.2 implies that

$$\beta_{a, |u|}(\Delta_{[u]}) \neq 0 \text{ and } \beta_{b+1, |v|}(\Delta_{[v]}) \neq 0.$$

Now we focus on $\Delta_{[v]}$, which contains the facet F . By Theorem 3.4, $\Delta_{[F]}$ has a complement $\Delta_{[w]}$ in $\Delta_{[v]}$ with $\beta_{b, |w|}(\Delta_{[w]}) \neq 0$.

We show that $\Delta_{[w]}$ and $\Delta_{[u]}$ are complements in Δ . From the fact that $\Delta_{[w]}$ and $\Delta_{[F]}$ are complements in $\Delta_{[v]}$ we see

$$F \not\subseteq w \text{ and } w \cup F = v = v' \cup F.$$

But $v' \cap F = \emptyset$ so $w \supseteq v'$. We write $w = v' \cup w'$ where $w' \subset F$.

It is clear that $w \cup u = V(\Delta)$. If G is a facet of Δ and

$$G \subset (u \cap w) = (u' \cup F) \cap (v' \cup w') \subset (u' \cap v') \cup F$$

then $(G \setminus F) \subseteq u' \cap v'$. Now there is a facet H of Δ such that $(H \setminus F) \subseteq (G \setminus F)$ and $(H \setminus F)$ is a facet of Γ . But then, since $(H \setminus F) \subseteq u' \cap v'$, we have that $(H \setminus F)$ is a facet of both $\Gamma_{[u']}$ and $\Gamma_{[v']}$, which contradicts the fact that these two are complements in Γ .

So $\Delta_{[w]}$ and $\Delta_{[u]}$ have no facets in common, and are therefore complements in Δ . □

Remark 3.6. It must be noted that under the assumptions of Theorem 3.5, not every $\mathbf{m}_u \in \text{LCM}(I)$ with $\beta_{a, |u|}(\Delta_{[u]}) \neq 0$ has a complement \mathbf{m}_v with $\beta_{b, |v|}(\Delta_{[v]}) \neq 0$.

For example consider the ideal $I = (ab, bc, cd, de)$. The only complement of $m_u = bcd$ is $m_v = abde$. We have

$$\beta_{3,5}(S/I) \neq 0 \text{ and } \beta_{2,3}(S/(bc, cd)) \neq 0$$

but

$$\beta_{1,4}(S/(ab, de)) = 0.$$

Theorem 3.7 (Subadditivity of Betti numbers of forests). *Let Δ be a simplicial forest with facet ideal I and suppose i is at most the projective dimension of S/I , and $i = a + b$ where $a, b > 0$. Then $t_a(I) + t_b(I) \geq t_{a+b}(I)$.*

Proof. Assume without loss of generality that Δ has n vertices and $\beta_{i,n}(\Delta) \neq 0$. If Δ has only one facet, there is nothing to prove as $i = 1$. If Δ has more than one facet, by Theorem 3.5 there are complements $\Delta_{[u]}$ and $\Delta_{[v]}$ in Δ with $\beta_{a,|u|}(\Delta_{[u]}) \neq 0$ and $\beta_{b,|v|}(\Delta_{[v]}) \neq 0$. It follows that

$$t_a(I) + t_b(I) \geq |u| + |v| \geq n = t_{a+b}(I).$$

□

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